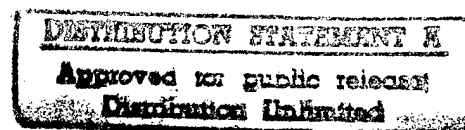# CONTRACT TECHNICAL REPORT

"A STRAIN-BASED FORMULATION FOR THE COUPLED VISCOELASTIC/DAMAGE BEHAVIOR"

By

Khaled Abdel-Tawab and Y. Jack Weitsman

Prepared for: Office of Naval Research
Arlington, Virginia

19980102 059

DTIC QUALITY INSPECTED 4

**Mechanical and Aerospace Engineering
and Engineering Science
THE UNIVERSITY OF TENNESSEE
Knoxville, TN   37996-2030**

UT

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
Unclassified	

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
	Unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Mechanical & Aerospace Engr. & Engr. Sci. University of Tennessee		Office of Naval Research

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
307 Perkins Hall Knoxville, TN 37996-2030	Office of Naval Research, code 432 800 N. Quincy Ave., BCT #1, Room 528 Arlington, VA 22217

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
Office of Naval Research		N00014-96-1-0821 (ONR Contract #)

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.			
	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
Arlington, VA 22217				

11. TITLE (Include Security Classification)
"A Strain-Based Formulation for the Coupled Viscoelastic/ Damage Behavior"

12. PERSONAL AUTHOR(S)
Khaled Abdel-Tawab and Y. Jack Weitsman

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM 2/1/97 TO 9/30/98	97-07-15	23

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A strain-based thermodynamics framework is proposed for modeling the continuum damage behavior of viscoelastic materials. Damage is represented by an internal state variable in the form of a symmetric second rank tensor. The effect of damage on the constitutive behavior is introduced through direct coupling between the damage variable and the viscoelastic internal state variables. This approach accounts for time-dependent damage as well as damage-induced changes in material symmetry. Also, damage evolution is modeled by employing the concept of damage surfaces.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION	
UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS ☐	Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Dr. Yapa Rajapakse	(703) 696-4405	

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

A STRAIN-BASED FORMULATION FOR THE COUPLED VISCOELASTIC/DAMAGE BEHAVIOR

Khaled Abdel-Tawab and Y. Jack Weitsman

Department of Mechanical and Aerospace Engineering and Engineering Science

The University of Tennessee, Knoxville, TN 37996

ABSTRACT

A strain-based thermodynamics framework is proposed for modeling the continuum damage behavior of viscoelastic materials. Damage is represented by an internal state variable in the form of a symmetric second rank tensor. The effect of damage on the constitutive behavior is introduced through direct coupling between the damage variable and the viscoelastic internal state variables. This approach accounts for time-dependent damage as well as damage-induced changes in material symmetry. Also, damage evolution is modeled by employing the concept of damage surfaces.

1

1. INTRODUCTION

The growing interest in the use of polymeric materials (e.g. plastics and polymeric composites) for structural applications mandates appropriate knowledge of the mechanical behavior as well as the durability of these materials. It is well-known that polymeric materials creep viscoelastically. In addition, experimental investigations (e.g. Suvorova, 1985; Schapery, 1989; Tuttle *et al.*, 1995; Smith and Weitsman, 1996) indicate that polymeric composites may undergo distributed damage in the form of a multitude of microcracks. The objective of this article is to establish a framework for the constitutive modeling of the foregoing features of material behavior. Such framework is essential for a reliable engineering design.

Up to the present time, most of the efforts for modeling distributed damage have been directed toward brittle materials exhibiting elastic behavior and metals exhibiting plastic or creep response (e.g. Krajcinovic, 1996). Less attention has been paid to the modeling of damage in viscoelastic materials. Notably, Schapery (1981, 1994, 1996a) established a basic formulation for viscoelastic response that is accompanied by microstructural changes, such as profuse microcracking. The microstructural changes are represented in Schapery's work by means of a set of internal state variables whose evolutionary laws are motivated by considerations of viscoelastic fracture mechanics. It may also be mentioned that Weitsman (1988) attempted to model the coupling between viscoelasticity and damage for a special class of linear viscoelastic materials.

In a recent article by the present authors (Abdel-Tawab and Weitsman, 1996) a stress-based formulation for modeling the coupling between viscoelasticity and distributed damage was developed and applied to a swirl-mat polymeric composite. The effect of damage on material behavior was introduced through the concept of effective stress. Also, damage evolution was related by the empirical Kachanov-Rabotnov forms (Kachanov, 1986), which are best suited for monotonic creep loadings. For more complex loading histories the concept of damage surfaces (Krajcinovic, 1996) offers a more versatile approach to damage evolution.

2

It is well-known that damage surfaces are better expressed in strain space than in stress space (Ju, 1989; Krajcinovic, 1996). This concept, which seems particularly appropriate for viscoelastic response – where creep occurs at all stress levels, provided a motivation for the present strain-based formulation of viscoelasticity coupled with damage. An additional motivation is the fact that strain-based viscoelastic constitutive models are more convenient for implementation into finite element codes than stress-based ones (e.g. ABAQUS, 1996). The present formulation employs concepts of continuum damage mechanics as well as several existing concepts of the thermodynamic theory of viscoelastic materials (Biot, 1954; Schapery, 1964). This format accounts for time-dependent damage as well as damage induced changes in material symmetry.

In Section 2 of this article, we present a general thermodynamics framework that accounts for both viscoelastic and damage processes. We proceed by modeling the coupling between these two processes in Section 3. In Section 4, damage evolution is modeled through the concept of damage surfaces and is illustrated by a simple example in Section 5. Section 6 concludes with a summary and some remarks pertinent to the present work.

2. THERMODYNAMICS FRAMEWORK

Consider a polymeric material and let γ_r $(r = 1, 2, \ldots, R)$ denote R scalar valued internal state variables representing the internal degrees of freedom of molecular motion in the polymeric chains. The internal state variable representing damage can be related in terms of tensorial quantities of even ranks, which can be associated with the spatial distributions of microcracks (Krajcinovic, 1996). For simplicity, the damage variable is chosen as a symmetric second rank tensor ω_{ij} with dimensionless components. This damage variable is capable of simulating changes in material symmetry such that an initially damage free isotropic material may become, at most, orthotropic upon damage formation (Cordebois and Sidoroff, 1982; Lemaitre, 1992). Despite the shortcomings of the abovementioned damage variable (e.g. Lubarda and

Krajcinovic, 1993), it was adopted by several workers in the field of damage mechanics (e.g. Cordebois and Sidoroff, 1982; Murakami and Imaizumi, 1982; Chen and Chow, 1995) due to its relative simplicity and applicability to practical circumstances. It should be mentioned that the present formulation can be readily modified to accommodate damage variables of other tensorial ranks. Throughout this article the subscripts r and q are reserved for scalar quantities; and $a, b, c, d, i, j, k, l, m$ and n are associated with tensorial quantities and cover the range $1, 2, 3$. Also, the summation convention is implied over the range of repeated indices unless stated otherwise.

Viscoelasticity and damage are irreversible thermodynamic processes. For a closed system and small strains, the entropy production inequality can be written in the form (Coleman and Gurtin, 1967)

$$-\dot{\psi} + \sigma_{ij}\,\dot{\varepsilon}_{ij} - \mathcal{S}\,\dot{T} - \frac{h_i\,T_{,i}}{T} \geq 0 \,, \tag{1}$$

where ψ is the Helmholtz free energy (per unit volume), σ_{ij} – components of a suitably defined volume averaged stress tensor, ε_{ij} – components of the infinitesimal strain tensor, $\mathcal{S}$ – entropy (per unit volume), T – temperature, h_i – components of the heat flux vector, $T_{,i} = \partial T/\partial x_i$ – components of the temperature gradient, and x_i – space coordinates. Also, in (1) an overdot signifies differentiation with respect to time.

Consider a Helmholtz free energy of the form

$$\psi = \psi(\varepsilon_{ij}, \gamma_r, \omega_{ab}, T) \,. \tag{2}$$

The function ψ is assumed to be continuous and sufficiently differentiable with respect to its arguments. Considerations of the entropy production inequality in (1) together with the functional dependence in (2) give the following familiar relations

$$\sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}} \,, \tag{3}$$

$$\mathcal{S} = -\frac{\partial \psi}{\partial T} \,, \tag{4}$$

4

and

$$\Gamma_r \, \dot{\gamma}_r + \Omega_{ab} \, \dot{\omega}_{ab} - \frac{h_i \, T_{,i}}{T} \geq 0 \, , \tag{5}$$

where Γ_r and Ω_{ab} are the thermodynamic forces conjugate to the internal state variables γ_r and ω_{ab}, respectively, and are given by

$$\Gamma_r = -\frac{\partial \psi}{\partial \gamma_r} \, , \tag{6}$$

and

$$\Omega_{ab} = -\frac{\partial \psi}{\partial \omega_{ab}} \, . \tag{7}$$

Finally, from the dissipation inequality (5) we have the following requirements

$$\Gamma_r \, \dot{\gamma}_r \geq 0 \, , \tag{8}$$

$$\Gamma_r \, \dot{\gamma}_r + \Omega_{ab} \, \dot{\omega}_{ab} \geq 0 \, . \tag{9}$$

Inequality (8) must be satisfied whenever viscoelastic deformation occurs, while when deformation is accompanied by damage inequality (9) should be satisfied as well.

3. CONSTITUTIVE MODELING

3.1. General Formulation

In this section a constitutive model is formulated for the case of linear viscoelastic behavior coupled with damage. For simplicity, attention is restricted to the case of isothermal behavior. The extension to the general case of nonisothermal conditions can be made following the same approach adopted here. The formulation will be first established for fixed strain ε_{ij} and damage ω_{ab} and subsequently extended to fluctuating ε_{ij} and ω_{ab}.

For fixed ε_{ij} and ω_{ab} an irreversible thermodynamic process is triggered in the material, which prompts the viscoelastic internal state variables γ_r to drift spontaneously toward their equilibrium values γ_r^e. Under isothermal conditions, all γ_r^e are

independent of temperature, hence

$$\gamma_r^e = \gamma_r^e(\varepsilon_{ij}, \omega_{ab}) \ . \tag{10}$$

These equilibrium values are assumed to be continuous and sufficiently differentiable functions of their arguments. Assuming that all γ_r and γ_r^e are sufficiently small, a Taylor series expansion for ψ about γ_r^e takes the form

$$\psi = \psi_e + \frac{1}{2}\, \psi_{rq}\, (\gamma_r - \gamma_r^e)\, (\gamma_q - \gamma_q^e) + \text{H.O.T.} \ , \tag{11}$$

where

$$\psi_e = \psi_e(\varepsilon_{ij}, \omega_{ab})$$

is the value of ψ at equilibrium,

$$\psi_{rq} = \left(\frac{\partial^2 \psi}{\partial \gamma_r \partial \gamma_q} \right)_e$$

is a symmetric matrix considered to be constant, and H.O.T. refers to higher order terms neglected due to smallness of γ_r and γ_r^e. In the above relations, and in the sequel, the subscript " e " implies that a quantity is calculated at $\gamma_r = \gamma_r^e \quad \forall\, r$. Note that at equilibrium ψ is minimum (Callen, 1960; Prigogine, 1967), and hence

$$\left(\frac{\partial \psi}{\partial \gamma_r} \right)_e = 0 \ ,$$

and

$$\psi_{rq}\, \delta\gamma_r \delta\gamma_q \ > \ 0 \ .$$

Consequently, there is no linear term in (11) and ψ_{rq} is a positive definite matrix. It should be mentioned that an expansion similar to that in (11) was previously used by Lubliner (1972).

Employing the usual assumption of viscous-like resistance (Biot, 1954; Schapery, 1964), let

$$\Gamma_r = a_{rq}\, \dot{\gamma}_q \ , \tag{12}$$

6

where, according to Onsager's principle (Callen, 1960; Fung, 1965), a_{rq} is a symmetric matrix. Substitution of (12) into inequality (8) gives

$$a_{rq}\,\dot{\gamma}_r\,\dot{\gamma}_q \;\geq\; 0\,.$$

Hence, the matrix a_{rq} is positive semi-definite. Note that in the general case a_{rq} is a function of temperature, but since we are considering only isothermal conditions then a_{rq} is constant.

Equations (6), (11) and (12) yield

$$a_{rq}\,\dot{\gamma}_q + \psi_{rq}\,\gamma_q = \psi_{rq}\,\gamma_q^e\,. \qquad (13)$$

Since a_{rq} is a constant symmetric positive semi-definite matrix and ψ_{rq} is a constant symmetric positive definite matrix, it is possible to rewrite (13) in a diagonalized form (Meirovitch,1967) as

$$A_r\,\dot{\hat{\gamma}}_r + \Psi_r\,\hat{\gamma}_r = \Psi_r\,\hat{\gamma}_r^e \qquad (\text{no sum over } r)\,, \qquad (14)$$

where $\hat{\gamma}_r$ are transformed internal state variables, each being a linear combination of the original internal state variables γ_q. The parameters $\hat{\gamma}_r^e$ are the equilibrium values corresponding to $\hat{\gamma}_r$ and are obtained from γ_q^e by the same linear transformation as that for $\hat{\gamma}_r$. Also, A_r and Ψ_r are constants such that $A_r \geq 0$ and $\Psi_r > 0$.

For fixed strain and damage, the solution of equation (14) is

$$\hat{\gamma}_r = \hat{\gamma}_r^e \left(1 - e^{-t/\tau_r}\right) \qquad (\text{no sum over } r)\,, \qquad (15)$$

where τ_r are relaxation times given by

$$\tau_r = \frac{A_r}{\Psi_r} \qquad (\text{no sum over } r)\,. \qquad (16)$$

In terms of the transformed internal state variables, expansion (11) can be rewritten as

$$\psi = \psi_e + \frac{1}{2}\sum_r \Psi_r\,(\hat{\gamma}_r - \hat{\gamma}_r^e)^2 + \text{H.O.T.} \qquad (17)$$

7

The viscoelastic strain can now be obtained by substituting (17) into (3) bearing in mind that γ_r, and hence $\hat{\gamma}_r$, are to be kept fixed during the partial differentiation indicated in (3). Employing (15) we then obtain

$$\sigma_{ij} = \frac{\partial \psi_e}{\partial \varepsilon_{ij}} + \sum_r \frac{\partial \Lambda_r}{\partial \varepsilon_{ij}} e^{-t/\tau_r} , \qquad (18)$$

where

$$\Lambda_r = \Lambda_r(\varepsilon_{ij}, \omega_{ab}) = \frac{1}{2} \Psi_r (\hat{\gamma}_r^e)^2 \qquad \text{(no sum over } r\text{)} . \qquad (19)$$

The first term on the right hand side of (18) represents the long-term (rubbery) part of the behavior, and the second term represents the transient (time-dependent) part.

Motivated by previous works on linear elasticity with damage (e.g. Lemaitre and Chaboche, 1985; Lemaitre, 1992), we now recast the formulation in a format that retains a linear viscoelastic relaxation modulus and introduces the effects of damage by mapping the stress and strain into "damage effective" stress and strain, respectively. To this end, consider first the transient part in (18). Expanding Λ_r in terms of strain around the reference state, $\varepsilon_{ij}^{\text{ref}} = 0$, up to quadratic terms to retain linearity one obtains

$$\Lambda_r = \frac{1}{2} \left(\frac{\partial^2 \Lambda_r}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \right)_0 \varepsilon_{ij} \varepsilon_{kl} , \qquad (20)$$

where the subscript 0 implies that a quantity is calculated at the reference state. Note that the constant term in (20) vanishes since $(\gamma_r^e)_0 = 0$, and hence $\hat{\gamma}_r^e = \Lambda_r = 0$ at the reference state. In addition, the linear term in (20) is discarded since it corresponds to a residual stress at the reference state, which is disregarded in the present formulation.

A more specific functional form for Λ_r can be obtained by realizing that the internal molecular motions represented by γ_r occur on a much smaller dimensional scale than that of damage represented by ω_{ab}. This suggests that all γ_r^e, and hence all $\hat{\gamma}_r^e$ and Λ_r, are likely to be affected by damage in a common manner; i.e. they have common dependence on ω_{ab}. Consequently, we can rewrite (20) in the form

$$\Lambda_r = \frac{1}{2} P_{ijab} \Delta C_{abcd}^r P_{cdkl} \varepsilon_{ij} \varepsilon_{kl} \qquad \forall r . \qquad (21)$$

8

In (21), ΔC_{ijkl}^r is a double symmetric fourth rank tensor (i.e. $\Delta C_{ijkl}^r = \Delta C_{jikl}^r = \Delta C_{ijlk}^r = \Delta C_{klij}^r$), and $P_{ijkl} = P_{ijkl}(\omega_{ab})$ is a double symmetric fourth rank tensor valued function of the damage variable ω_{ab} such that

$$\text{at} \quad \omega_{ab} = 0 \quad \longrightarrow \quad P_{ijkl} = I_{ijkl} \,, \tag{22}$$

where

$$I_{ijkl} = \frac{1}{2} \left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk} \right) ,$$

is the unit fourth rank tensor and δ_{ij} is Kronecker delta.

Using (21), the transient part of (18) takes the form

$$\sum_r \frac{\partial \Lambda_r}{\partial \varepsilon_{ij}} \, e^{-t/\tau_r} = P_{ijab} \, \Delta C_{abcd}(t) \, P_{cdkl} \, \varepsilon_{kl} \,, \tag{23}$$

where

$$\Delta C_{ijkl}(t) = \sum_r \Delta C_{ijkl}^r \, e^{-t/\tau_r} \,. \tag{24}$$

From (18), (22) and (23) it is clear that ΔC_{ijkl} is the undamaged transient (time-dependent) stiffness tensor.

Consider now the long-term part of (18). The equilibrium Helmholtz free energy ψ_e can be expanded around the reference state in the form

$$\psi_e = \frac{1}{2} \left(\frac{\partial^2 \psi_e}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \right)_0 \varepsilon_{ij}\, \varepsilon_{kl} \,. \tag{25}$$

Following common practice in continuum damage mechanics (e.g. Lemaitre and Chaboche 1985; Lemaitre, 1992), we further assume that ψ_e depends on damage in the separable form

$$\psi_e = \frac{1}{2} \, Q_{ijab} \, C_{abcd}^e \, Q_{cdkl} \, \varepsilon_{ij}\, \varepsilon_{kl} \,, \tag{26}$$

where $Q_{ijkl} = Q_{ijkl}(\omega_{ab})$ is a double symmetric fourth rank tensor valued function of ω_{ab} such that $Q_{ijkl} = I_{ijkl}$ at $\omega_{ab} = 0$, and consequently C_{ijkl}^e is the undamaged long-term (rubbery) stiffness tensor which is typically positive definite (Fung, 1965). Using (26), the long-term part of (18) takes the form

$$\frac{\partial \psi_e}{\partial \varepsilon_{ij}} = Q_{ijab} \, C_{abcd}^e \, Q_{cdkl} \, \varepsilon_{kl} \,, \tag{27}$$

9

Relations (23) and (27) indicate that the long-term part of the behavior can in general depend on damage in a manner that differs from that of the transient part. However, for simplicity, we assume here that both parts have the same dependence on damage so that

$$Q_{ijkl} = P_{ijkl} \, . \tag{28}$$

Assuming that the inverse P_{ijkl}^{-1} exists, define the following "damage effective" stress and strain tensors

$$\tilde{\sigma}_{ij} = P_{klij}^{-1} \, \sigma_{kl} \, . \tag{29}$$

$$\tilde{\varepsilon}_{ij} = P_{ijkl} \, \varepsilon_{kl} \, , \tag{30}$$

then relation (18) can be rewritten in the compact form

$$\tilde{\sigma}_{ij} = C_{ijkl}(t) \, \tilde{\varepsilon}_{kl} \, , \tag{31}$$

where

$$C_{ijkl}(t) = C_{ijkl}^{\mathrm{e}} + \Delta C_{ijkl}(t) \, , \tag{32}$$

is the overall (long-term and transient) stiffness tensor of the undamaged material. Relations (29) and (30) are consistent with the formulations of the concepts of effective stress and effective strain (e.g. Cordebois and Sidoroff, 1982; Simo and Ju, 1987), where the mapping tensor for the effective stress is taken to be the inverse of that of the effective strain. Also, note that both $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are symmetric due to the hypothesized double-symmetry of P_{ijkl}.

Relation (31) suggests that for a given damage level ω_{ab}, instantaneous mapping of the actual stress σ_{kl} and strain ε_{kl} according to (29) and (30), respectively, lead to new stress $\tilde{\sigma}_{ij}$ and strain $\tilde{\varepsilon}_{ij}$ quantities that are related by the usual linear viscoelastic constitutive relation for fixed strain (e.g. Fung, 1965). Upon hypothesizing time-translation invariance, and since $\tilde{\sigma}_{ij}$ is linear in $\tilde{\varepsilon}_{kl}$, a straightforward application of the superposition principle (Pipkin, 1986) to expression (31) yields

$$\tilde{\sigma}_{ij} = \int_{0^-}^{t} C_{ijkl}(t - \tau) \frac{d\tilde{\varepsilon}_{kl}}{d\tau} \, d\tau \, . \tag{33}$$

10

Allowing for spatial variations of stress and damage the total derivative $d/d\tau$ inside the integral is replaced by a partial derivative $\partial/\partial\tau$, holding the spatial coordinates x_i fixed. Thus

$$\tilde{\sigma}_{ij} = \int_{0^-}^{t} C_{ijkl}(t - \tau)\frac{\partial \tilde{\varepsilon}_{kl}}{\partial \tau} d\tau \ . \tag{34}$$

Equation (34) is the stress-strain constitutive relation for the coupled linear viscoelastic/damage behavior, and can be expressed in terms of the actual stress and strain as

$$\sigma_{ij} = P_{ijab} \int_{0^-}^{t} C_{abcd}(t - \tau)\frac{\partial (P_{cdkl}\,\varepsilon_{kl})}{\partial \tau} d\tau \ . \tag{35}$$

It should be noted that at $t = 0$

$$C_{ijkl}(0) = C_{ijkl}^{\mathrm{o}} = C_{ijkl}^{\mathrm{e}} + \sum_{r} \Delta C_{ijkl}^{r} \ , \tag{36}$$

where C_{ijkl}^{o} is the initial (elastic) stiffness tensor which is positive definite (Fung, 1965). For the special case of isotropic virgin material response, the overall stiffness tensor C_{ijkl} takes the form (Fung, 1965)

$$C_{ijkl}(t) = 2\,G(t)\,I_{ijkl} + \left[K(t) - \frac{2}{3}\,G(t)\right]\delta_{ij}\,\delta_{kl} \ , \tag{37}$$

where $G(t)$ is the overall shear modulus and $K(t)$ is the overall bulk modulus given, respectively, by

$$G(t) = G_{\mathrm{o}} + \Delta G(t) \ , \tag{38}$$

and

$$K(t) = K_{\mathrm{o}} + \Delta K(t) \ . \tag{39}$$

In the above expressions G_{o} and K_{o} are the instantaneous shear and bulk moduli, respectively; and $\Delta G(t)$ and $\Delta K(t)$ are the transient shear and bulk moduli, respectively, obtained from (24) as

$$\Delta G(t) = \sum_{r} \Delta G_{r} e^{-t/\tau_r} \ , \tag{40}$$

and

$$\Delta K(t) = \sum_{r} \Delta K_{r} e^{-t/\tau_r} \ , \tag{41}$$

11

where ΔG_r and ΔK_r are positive constants.

3.2. The Dissipation Inequality

The thermodynamic force Ω_{ab} conjugate to ω_{ab} can be obtained by substituting (17) into (7) using (21), (26) and (28)

$$\Omega_{ab} = -P_{ijcd}\,\widehat{C}_{cdmn}\,\frac{\partial P_{mnkl}}{\partial \omega_{ab}}\,\varepsilon_{ij}\,\varepsilon_{kl}\,, \tag{42}$$

where $\widehat{C}_{ijkl}$ is given by

$$\widehat{C}_{ijkl} = C^{\mathrm{o}}_{ijkl} - \sum_r \left(\frac{\hat{\gamma}_r}{\hat{\gamma}^{\mathrm{e}}_r}\right)\Delta C^r_{ijkl}\,. \tag{43}$$

Since

$$0 \le \left(\frac{\hat{\gamma}_r}{\hat{\gamma}^{\mathrm{e}}_r}\right) \le 1\,; \quad \forall\, r\,,$$

then $\widehat{C}_{ijkl}$ is bounded by C^{e}_{ijkl} and C^{o}_{ijkl} corresponding, respectively, to the upper and lower limits of $\hat{\gamma}_r/\hat{\gamma}^{\mathrm{e}}_r$. Since C^{e}_{ijkl} and C^{o}_{ijkl} are positive definite then it follows that $\widehat{C}_{ijkl}$ is also positive definite.

Employing (6) and (7), the dissipation inequality (9) can be expressed as

$$\sum_r A_r\,\dot{\hat{\gamma}}^2_r - P_{ijab}\,\widehat{C}_{abcd}\,\dot{P}_{cdkl}\,\varepsilon_{ij}\,\varepsilon_{kl} \ge 0\,, \tag{44}$$

where

$$\dot{P}_{cdkl} = \frac{\partial P_{cdkl}}{\partial \omega_{mn}}\,\dot{\omega}_{mn}\,. \tag{45}$$

Noting that the first term on the left-hand side of (44) is always non-negative, then a sufficient but not necessary condition to satisfy (44) is

$$-P_{ijab}\,\widehat{C}_{abcd}\,\dot{P}_{cdkl} \quad \longrightarrow \quad \text{positive semi} - \text{definite}\,. \tag{46}$$

3.3. The Mapping Tensor

The functional form of the mapping tensor P_{ijkl} is restricted by the requirement that P_{ijkl} is double symmetric in addition to the requirements in (22) and (44). In

12

general, P_{ijkl} is an anisotropic fourth rank tensor function of ω_{ab}. However, due to the complexity of anisotropic functional forms (Zheng, 1994) and the fact that damage-induced anisotropy (or, more precisely, orthotropy) can be deduced from a symmetric second rank damage tensor, then the more complex anisotropic functional form may be avoided.

Following Murakami and Imaizumi (1982), a simpler representation of P_{ijkl} can be obtained by taking it as an isotropic fourth rank tensor function of ω_{ab}. A further simplification of the representation of P_{ijkl} is obtained by considering a case of dilute concentration of microcracks in which P_{ijkl} is linear in ω_{ab}. In this case, P_{ijkl} can be written as (Murakami and Imaizumi, 1982)

$$P_{ijkl} = c_1\,\delta_{ij}\,\delta_{kl} + c_2\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) + c_3\,\delta_{ij}\,\omega_{kl} + c_4\,\delta_{kl}\,\omega_{ij}$$

$$+\ c_5\,(\delta_{ik}\,\omega_{jl} + \delta_{il}\,\omega_{jk} + \delta_{jk}\,\omega_{il} + \delta_{jl}\,\omega_{ik}) + \text{H.O.T.}\,, \tag{47}$$

where c_α $(\alpha = 1, 2, \ldots, 5)$ are constants.

From (47), it is clear that double symmetry of P_{ijkl} dictates that $c_3 = c_4$. Also, the requirement in (22) renders $c_1 = 0$ and $c_2 = 1/2$. To determine $c_3 = c_4$ and c_5, we consider the special case of isotropic damage in which the ensuing damage pattern does not affect the symmetry of the virgin material. In this case damage is represented by a single scalar ω so that

$$\omega_{ab} = \omega\,\delta_{ab} \quad ; \quad 0 \le \omega < 1\,, \tag{48}$$

and the mapping tensor P_{ijkl} takes the form (Simo and Ju, 1987)

$$P_{ijkl} = (1 - \omega)\,I_{ijkl}\,, \tag{49}$$

which is the inverse of the corresponding mapping tensor that maps the applied stress into the Kachanov effective stress in the case of scalar damage (e.g. Kachanov, 1986). The functional form in (49) can be recovered from (47) by setting $c_3 = c_4 = 0$ and taking $c_5 = -1/4$. Thus, the simplest possible form of P_{ijkl} becomes

$$P_{ijkl} = I_{ijkl} - \frac{1}{4}\,(\delta_{ik}\,\omega_{jl} + \delta_{il}\,\omega_{jk} + \delta_{jk}\,\omega_{il} + \delta_{jl}\,\omega_{ik})\,. \tag{50}$$

13

It is interesting to note that P_{ijkl}^{-1} obtained from (50) indeed coincides with one of the forms proposed by Chen and Chow (1995) for the tensor that maps the applied stress into an effective stress.

The complete formulation of the constitutive model requires an expression for the evolution of the damage tensor $\dot{\omega}_{ab}$ such that (44) is satisfied. Such an expression can be formally derived from thermodynamic considerations (e.g. Lemaitre and Chaboche, 1985; Lemaitre, 1992), but the usefulness of such approach seems to be restricted to elastic response with damage. In practice, the form of the damage evolution equation depends on the material considered and the applied loading. This dependence is better correlated within the concept of damage surfaces (Krajcinovic, 1996) as discussed in the following section.

4. DAMAGE EVOLUTION

The approach adopted here for describing damage evolution follows closely that presented by Simo and Ju (1987) and Lubarda and Krajcinovic (1995). This approach has two main ingredients. First, a damage surface is introduced in strain space to distinguish between the material states associated with evolving damage and those with stationary damage. Second, a damage potential is assumed to exist, from which the constitutive law of damage growth (i.e. the damage rate $\dot{\omega}_{ab}$) can be derived.

To characterize damage evolution, i.e. *damage loading* conditions, a damage function $f(\varepsilon_{ij}, \kappa)$ is introduced so that

$$f(\varepsilon_{ij}, \kappa) \leq 0 \,, \tag{51}$$

where κ is a positive scalar damage threshold history parameter and at the initial onset of damage $\kappa = \kappa_o$. The equality in (51), i.e. $f = 0$, corresponds to strain states that lie on the damage surface and for which damage can evolve. For simplicity, the function f is chosen in the simple isotropic hardening form

$$f(\varepsilon_{ij}, \kappa) = \mathcal{F}(\varepsilon_{ij}) - \kappa \,, \tag{52}$$

14

where $\mathcal{F}$ is a scalar function of the strain.

Introduce a monotonic scalar function $\mathcal{G}(\Omega_{ab})$ such that the damage rate can be expressed as

$$\dot{\omega}_{ab} = \dot{\lambda}\,\frac{\partial \mathcal{G}}{\partial \Omega_{ab}}\,. \tag{53}$$

where λ is a monotonically increasing positive scalar, i.e.

$$\dot{\lambda} \geq 0\,. \tag{54}$$

Physically, λ represents a measure of the cumulative damage at the considered instant of the deformation process. The function $\mathcal{G}$ is referred to as the "damage potential".

Following Simo and Ju (1987) let

$$\dot{\lambda} = \dot{\kappa}\,, \tag{55}$$

and define *damage loading/unloading* conditions according to relations (51) and (54) together with

$$\dot{\lambda}\,f = 0\,. \tag{56}$$

Thus, if $f < 0$ then $\dot{\lambda} = 0$ and from (53) no damage evolution takes place, i.e. the so-called *damage unloading* from the current state of strain on the damage surface takes place. If $f = 0$ and $\dot{\lambda} = 0$ then *damage neutral loading* occurs. Finally, if $\dot{\lambda} \neq 0$ then $f = 0$ and *damage loading* takes place.

During *damage loading*, the consistency condition

$$\dot{f} = 0\,, \tag{57}$$

must always be satisfied. From (52) and (57) we have

$$\dot{\kappa} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}}\,\dot{\varepsilon}_{ij}\,. \tag{58}$$

Assuming that no damage healing occurs, i.e. the damage surface can only expand, then κ is obtained from (52) and (55) as

$$\kappa = \max\left\{\kappa_o, \mathcal{F}_{\max}\right\}\,, \tag{59}$$

15

where $\mathcal{F}_{\max}$ is the maximum value of $\mathcal{F}$ over the entire loading history. Substitution of (58) and (55) into (53) yields

$$\dot{\omega}_{ab} = \frac{\partial \mathcal{G}}{\partial \Omega_{ab}} \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} . \tag{60}$$

Thus, specification of the functional forms of $\mathcal{F}$ and $\mathcal{G}$ completes the formulation for the damage evolution. In practice, these functional forms depend on the material considered and the ensuing damage pattern. Example functional forms will be presented in the following section.

In the damage evolution equation (60) the thermodynamic force Ω_{ab} is given by expression (42) in which $\widehat{C}_{ijkl}$ is given by (43). Thus an explicit expression for Ω_{ab} requires evaluation of the ratio $\hat{\gamma}_r / \hat{\gamma}_r^e$. This ratio can be determined from the differential equation

$$\frac{d}{dt}\left(\frac{\hat{\gamma}_r}{\hat{\gamma}_r^e}\right) + \left(\frac{1}{\tau_r} + \frac{1}{2\,\Lambda_r}\frac{d\,\Lambda_r}{dt}\right)\left(\frac{\hat{\gamma}_r}{\hat{\gamma}_r^e}\right) = \frac{1}{\tau_r} \qquad \text{(no sum over } r\text{)}, \tag{61}$$

where Λ_r is given by (21). Equation (61) is obtained after simple algebraic manipulations of equation (14) and making use of (16) and (19).

It should be noted that Ω_{cd} depends on ω_{ab} explicitly through P_{mnkl} and also implicitly through the ratio $\hat{\gamma}_r / \hat{\gamma}_r^e$. Thus, in practice, equations (34), (60) and (61) need be implemented incrementally where for given strain and time increments an iterative procedure is required for determining the corresponding damage increment.

5. ILLUSTRATIVE EXAMPLE FOR THE DAMAGE FUNCTIONS

Two scalar functions $\mathcal{F}(\varepsilon_{ij})$ and $\mathcal{G}(\Omega_{ab})$ are needed in (60) to obtain an explicit damage evolution relation. The simplest possible representation of these functions is to take each as an isotropic function of its tensor argument. Thus

$$\mathcal{F} = \mathcal{F}(I_1^\varepsilon, I_2^\varepsilon, I_3^\varepsilon) , \tag{62}$$

and

$$\mathcal{G} = \mathcal{G}(I_1^\Omega, I_2^\Omega, I_3^\Omega) , \tag{63}$$

16

where I_1^β, I_2^β and I_3^β are the isotropic invariants of the corresponding second rank symmetric tensor β_{ij}. These isotropic invariants can be written as (Zheng, 1994)

$$I_1^\beta = \beta_{kk} \, ,$$

$$I_2^\beta = \sqrt{\beta_{ij}' \, \beta_{ij}'} \, ,$$

and

$$I_3^\beta = \det[\beta_{ij}] \, ,$$

where β_{ij}' is the deviatoric part of β_{ij}

$$\beta_{ij}' = \beta_{ij} - \frac{1}{3} \, \beta_{kk} \, \delta_{ij} \, .$$

To simplify matters, discard the dependence of $\mathcal{F}$ on I_3^ε – this is a customary constitutive assumption in damage modeling (Krajcinovic, 1996). Further, $\mathcal{F}$ is expressed in the following simple form (Lubarda and Krajcinovic, 1995)

$$\mathcal{F} = F_1 \, \varepsilon_{ij}' \, \varepsilon_{ij}' + F_2 \, \varepsilon_{kk}^2 \, , \qquad (64)$$

where F_1 and F_2 are constants.

Considering the case of isotropic damage, P_{ijkl} can be taken in the form (49) and (60) should reduce to the simple form

$$\dot{\omega}_{ab} = \dot{\omega} \, \delta_{ab} \, . \qquad (65)$$

It follows that $\mathcal{G}$ can depend only on I_1^Ω, i.e. on the trace of Ω_{ab}, so that

$$\frac{\partial \mathcal{G}}{\partial \Omega_{ab}} \sim \delta_{ab} \, .$$

Consider the case of a dilute concentration of microcracks where the interaction between microcracks as well as the effect of accumulated microcracks on further microcrack formation can be neglected. In this case, the rate of damage evolution $\dot{\omega}$ may be taken to be independent of the accumulated damage ω. Dependence of

17

$\dot{\omega}$ on ω, however, is implicit in Ω_{cc}. Thus, to eliminate dependence of $\dot{\omega}$ on ω the function $\mathcal{G}$ is taken to be linear in Ω_{cc}

$$\mathcal{G} = G\,\Omega_{cc}\,, \tag{66}$$

where G is a constant. Expression (66) can be thought of as the first term in a Taylor series expansion of $\mathcal{G}$ (the constant term in such expansion is immaterial to the present formulation). Thus addition of higher order terms in the expansion introduces dependence of $\dot{\omega}$ on ω.

Under *damage loading* conditions $\dot{\omega} > 0$ and the left hand side of the dissipation inequality (44) becomes

$$\sum_r A_r\,\dot{\gamma}_r^2 + (1-\omega)\,\dot{\omega}\,\widehat{C}_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}\,,$$

which is always positive since $\widehat{C}_{ijkl}$ is positive definite. Thus the requirement of positive dissipation is identically satisfied. Substitution of (64) and (66) into (60) and use of (65) yield damage evolution in the form

$$\dot{\omega} = \alpha\,\varepsilon'_{ij}\,\dot{\varepsilon}'_{ij} + \beta\,\varepsilon_{kk}\,\dot{\varepsilon}_{kk}\,, \tag{67}$$

where

$$\alpha = 2\,F_1\,G \qquad \text{and} \qquad \beta = 2\,F_2\,G$$

are free parameters that need to be determined from the damage evolution pattern in a considered problem. The first term on the right hand side of (67) represents the effect of the deviatoric part of the behavior on damage evolution, whereas the second term represents the effect of the hydrostatic part.

6. CONCLUDING REMARKS

In this article a thermodynamically consistent framework was proposed to model the coupling between linear viscoelastic deformation and microcrack damage. The effect of damage was incorporated into the constitutive equations in a form consistent with

18

the well-known effective stress and effective strain concepts, and damage evolution was related by the concept of damage surfaces.

Several tensorial ranks for the damage variable can be employed in the context of the present formulation. However, for simplicity, in this article damage is represented by a symmetric second rank tensor. This representation is capable of simulating some changes in material symmetry induced by microcrack damage. A more general representation is to take the damage variable in the form of a double symmetric fourth rank tensor ω_{abcd}. Such a representation is capable of simulating general damage-induced changes in material symmetry (Krajcinovic, 1996). This, however, complicates construction of the functional forms for the mapping tensor P_{ijkl} and the damage potential $\mathcal{G}$; since both have to be functions of double symmetric fourth rank tensors (ω_{abcd} in the case of P_{ijkl} and the conjugate thermodynamic force Ω_{abcd} in the case of $\mathcal{G}$). For fourth rank tensors, the definite forms of the integrity bases and invariants are not yet well established (Zheng, 1994).

An important remark is that in the present formulation the thermodynamic force conjugate to damage depends on the viscoelastic internal state variables as can be seen from relations (42) and (43). This differs from previous formulations by Schapery (1981, 1996a,b), where the thermodynamic force conjugate to damage is taken to be independent of the viscoelastic internal state variables and to depend only on the elastic (instantaneous) part of the deformation. The argument put forth by Schapery is that based on a viscoelastic fracture mechanics analysis (Schapery, 1984) it was found that the driving force for *existing* cracks is independent of the viscoelastic internal state variables. However, damage evolution occurs not only by the extension of existing microcracks, but also by the nucleation of new microcracks. In this general case, it is expected that the state of the viscoelastic deformation in the material should have a direct effect on the formation of new microcracks. Hence, the thermodynamic force conjugate to damage is expected to depend on the viscoelastic internal state variables. This is also in agreement with the internal state variable formulation for the coupled elastoplastic-damage behavior (e.g. Ju, 1989; Hansen

19

and Schreyer, 1994).

REFERENCES

ABAQUS (1996), *Theory Manual – Version 5.5*, HKS Inc., Pawtucket, Rhode Island.

Abdel-Tawab, K. and Weitsman, Y. (1996) "A Coupled Viscoelasticity/Damage Model with Application to Swirl-Mat Composites," Accepted for publication, *Int. J. Damage Mech.*

Biot, M.A. (1954) "Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena," *J. Appl. Phys.* , Vol. 25, No. 6, pp. 1385-1391.

Callen, H.B. (1960) *Thermodynamics*, John Wiley, New York.

Chen, X.F. and Chow C.L. (1995) "On Damage Strain Energy Release Rate Y," *Int. J. Damage Mech.*, Vol. 4, pp. 251-263.

Coleman, B.D. and Gurtin, M. (1967) "Thermodynamics with Internal Variables," *J. Chem. Phys.*, Vol. 47, pp. 597-613.

Cordebois, J.P. and Sidoroff, F. (1982) "Damage Induced Elastic Anisotropy," *Mechanical Behavior of Anisotropic Solids, Proc. EUROMECH Colloque 115,*

June 1979 (J.P. Boehler, Ed.), pp. 761-774, Martinus Nijhoff, The Netherlands.

Fung, Y.C. (1965) *Foundations of Solid Mechanics*, Prentice Hall, Englewood Cliffs, New Jersey.

Hansen, N.R. and Schreyer, H.L. (1994) "A Thermodynamically Consistent Framework For Theories of Elastoplasticity Coupled With Damage," *Int. J. Solids Struct.*, Vol. 31, No. 3, pp. 359-389.

Ju, J.W. (1989) "On Energy-Based Coupled Elastoplastic Damage Theories: Constitutive Modeling and Computational Aspects," *Int. J. Solids Struct.*, Vol. 25, No. 7, pp. 803-833.

Ju, J.W. (1990) "Isotropic and Anisotropic Damage Variables in Continuum Damage Mechanics," *J. Engng. Mech.*, ASCE, Vol. 116, No. 12, pp. 2764-2770.

Kachanov, L.M. (1986) *Introduction to Continuum Damage Mechanics*, Martinus Nijhoff, Dordrecht, The Netherlands.

Krajcinovic, D. (1996) *Damage Mechanics*, Elsevier, New York.

Lemaitre, J. and Chaboche, J.-L. (1985) *Mechanics of solid materials*, English translation, Cambridge Univ. Press, New York.

Lemaitre, J. (1992) *A Course on Damage Mechanics*, Springer-Verlag, New York.

Lubarda, V.A. and Krajcinovic, D. (1993) "Damage Tensors and the Crack Density Distributions," *Int. J. Solids Strcut.*, Vol. 30, pp. 2859-2877.

Lubarda, V.A. and Krajcinovic, D. (1995) "Constitutive Structure of Rate Theory of Damage in Brittle Elastic Solids," *Appl. Math. and Computations*, Vol. 67, pp. 81-101.

Lubliner, J. (1972) "On the Thermodynamic Foundation of Non-Linear Solid Mechanics," *Int. J. Non-Linear Solid Mech.*, Vol. 7, pp. 237-254.

21

Meirovitch, L. (1967) *Analytical Methods in Vibrations*, Macmillan, New York.

Murakami, S. and Imaizumi, T. (1982) "Mechanical Description of Creep Damage State and its Experimental Verification," *J. de Mécanique Théorique et Appliquée*, Vol. 1, No. 5, pp. 743-761.

Pipkin, A.C. (1986) *Lectures on Viscoelasticity Theory*, 2nd Ed., Applied Mathematical Sciences, Vol. 7, Springer-Verlag, New York.

Prigogine, I. (1967) *Thermodynamics of Irreversible Processes*, 3rd Ed., Interscience Publ., John Wiley & Sons Inc., New York.

Schapery, R.A. (1964) "Application of Thermodynamics to Thermomechanical, Fracture and Birefringent Phenomena in Viscoelastic Media," *J. Appl. Phys.*, Vol. 35, No. 5, pp. 1451-1465.

Schapery, R.A. (1981) " On Viscoelastic Deformation and Failure Behavior of Composite Materials with Distributed Flaws," *Advances in Aerospace Structures and Materials* (S.S. Wang and W.J. Renton, Eds.), ASME Publ., New York, pp. 5-20.

Schapery, R.A. (1984) " Correspondence Principles and a Generalized J-Integral for Large Deformation and Fracture Analysis of Viscoelastic Media," *Int. J. Fract.*, Vol. 25, New York, pp. 195-223.

Schapery, R.A. (1989) " Mechanical Characterization and Analysis of Inelastic Composite Laminates With Growing Damage," *Mechanics of Composite Materials and Structures* (J.N. Reddy and J.L. Tapley, Eds.), ASME Publ., AMD Vol. 100, New York, pp. 1-9.

Schapery, R.A. (1994) " Nonlinear Viscoelastic Constitutive Equations for Composites Based on Work Potentials," *Appl. Mech. Rev.*, Vol. 47, No. 6, part 2, pp. S269-S275.

Schapery, R.A. (1996a) " Characterization of Nonlinear, Time-dependent Polymers and Polymeric Composites for Durability Analysis," *Progress in Durability Analysis of Composite Systems* (A.H. Cardon, H. Fukuda and K. Reifsnider, Eds.), Balkema Publ., Rotterdam, pp. 21-38.

Schapery, R.A. (1996b) " Nonlinear Constitutive Relations for Polymers and Polymeric Composites with Distributed Damage," Univ. of Texas Report No. SSM-96-5, September 1996.

Simo, J.C. and Ju, J.W. (1987) "Strain- and Stress-Based Continuum Damage Models–I. Formulation," *Int. J. Solids Struct.*, Vol. 23, pp. 821-840.

Smith, L.V. and Weitsman, Y. (1996) " The Visco-Damage Mechanical Response of Swirl-Mat Composites," Univ. of Tennessee Technical Report No. MAES96-2.0-CM, June, 1996.

Suvorova, J.V. (1985) "The Influence of Time and Temperature on the Reinforced Plastic Strength," in *Handbook of Composites*, Vol. 3–Failure Mechanics of Composites (G.C. Sih and A.M. Skudra, Eds.), Elsevier, New York, pp. 177-213.

Tuttle, M.E., Pasricha, A. and Emery, A.F. (1995) "The Nonlinear Viscoelastic-Viscoplastic Behavior of IM7/5260 Composites Subjected to Cyclic Loading," *J. Composite Mater.*, V. 29, No. 15, pp. 2025-2046.

Weitsman, Y. (1988) " A Continuum Damage Model for Viscoelastic Materials," *J. Appl. Mech.*, Vol. 55, pp. 773-780.

Zheng, Q-S. (1994) " Theory of Representation of Tensor Functions – A Unified Invariant Approach to Constitutive Equations," *Appl. Mech. Rev.*, Vol. 47, No. 11, pp. 545-586.